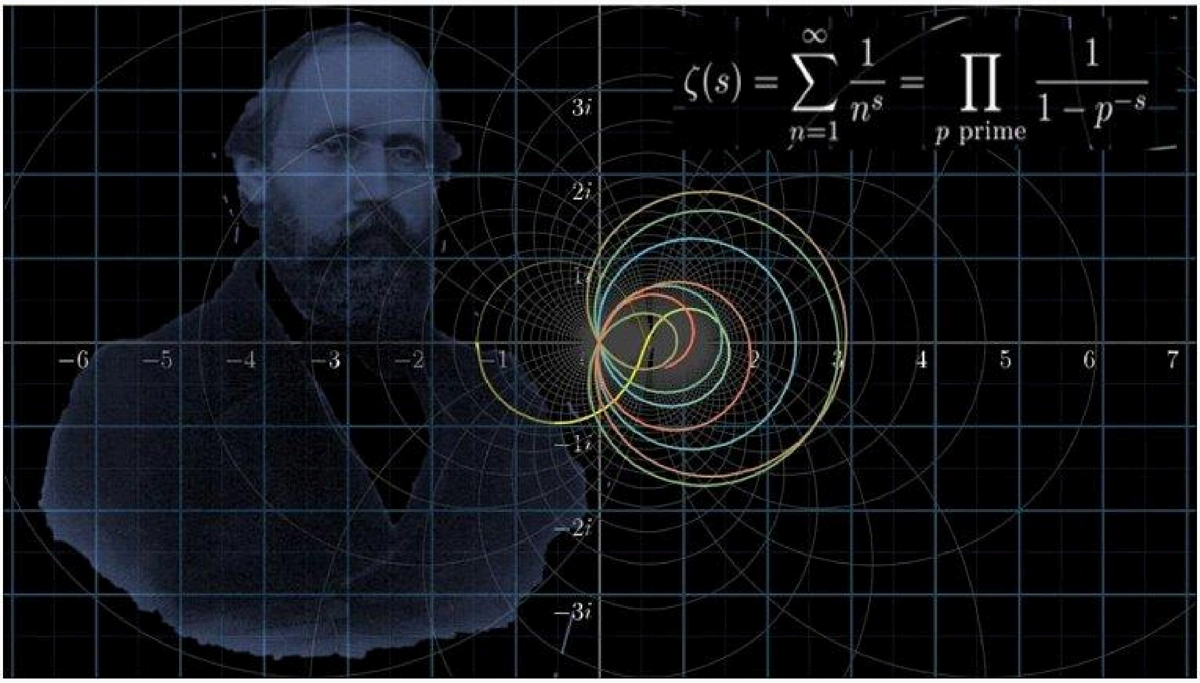
Evidence of equivalent conditions for the Riemann Hypothesis



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Abstrakt / Abstract:

Hľadanie dôkazov ekvivalentných podmienok pre RH (Riemannova hypotéza). Ekvivalentné podmienky boli vytvorené v minulosti autormi: Srinivasa Ramanujan (Ramanudžan), Lagarias, Gronwall ,Robin (viď. referencie (1),(2)).

Eng: Searching for evidence of equivalent conditions for RH (Riemann's hypothesis). Equivalent conditions have been created in the past by the authors: Srinivasa Ramanujan (Ramanujan), Lagarias, Gronwall, Robin (see references (1), (2)).

Úvod / Introduction

/SK: RH - Riemannova hypotéza /RH - Riemann hypothesis.
Táto publikácia je venovaná hľadaniu dôkazov v prospech Riemannovej hypotézy. Nadväzuje priamo na články referencia (7),(8),(9). Články referencia (7),(8),(9) obsahujú postupnosť mojich úvah ako nájsť dôkaz ekvivalentných podmienok pre RH, ako urobil v minulosti Srinivasa Ramanujan (Ramanudžan), Lagarias, Gronwall ,Robin (viď. referencie (1),(2)). Túto publikáciu treba vnímať v kontexte článkov - referencie (7),(8),(9). V publikácii budem postupne predkladať kroky, postupy, ktoré vedú na dôkaz ekvivalentných podmienok pre RH.
POZNÁMKA: log x=ln x, vo vzťahoch nižšie je log x - prirodzený logaritmus.

/Eng: This publication is devoted to finding evidence in favor of the Riemann hypothesis. It follows up directly on the articles with the reference (7), (8), (9). Articles with the reference (7), (8), (9) contain a sequence of my thoughts on how to find evidence of equivalent conditions for RH, as Srinivasa Ramanujan (Ramanujan), Lagarias, Gronwall, Robin did in the past (see references (1), (2).)). This publication should be read in the context of the articles - references (7), (8), (9). In the publication, I will gradually present the steps and procedures that lead to the proof of equivalent conditions for RH.

NOTE: log x = ln x, log x - the natural logarithm.

Referencie (1),(2),(5) / Reference (1),(2),(5).
citácia/ citation:
The sum-of-divisors function σ is defined by

$$\sigma(n) := \sum_{d|n} d$$

For example, σ(4) = 7 and σ(pn) = (p + 1)σ(n), if p is a prime not dividing n. In 1913, the Swedish mathematician Thomas Gronwall found the maximal order of σ.

Theorem 1. (Gronwall) The function

$$G(n) := \frac{\sigma(n)}{n \log(\log n)}$$

satisfies $\lim_{n \to \infty} \sup G(n) = e^\gamma = 1.78107...$, where γ is the Euler-Mascheroni constant.

Theorem 2. (Ramanujan) If the Riemann Hypothesis is true, then

$$G(n) < e^\gamma \quad (n \gg 1)$$

Here n ≫ 1 means for all sufficiently large n. In 1984, the French mathematician Guy Robin proved that a stronger statement about the function G is equivalent to the RH.

Theorem 3. (Robin) The Riemann Hypothesis is true if and only if

$$G(n) < e^\gamma ; \quad (n > 5040).$$

Theorem 4. (Lagarias) The Riemann Hypothesis is true if and only if

$\sigma(n) < Hn + exp(Hn)log(Hn)\ (n > 1)$

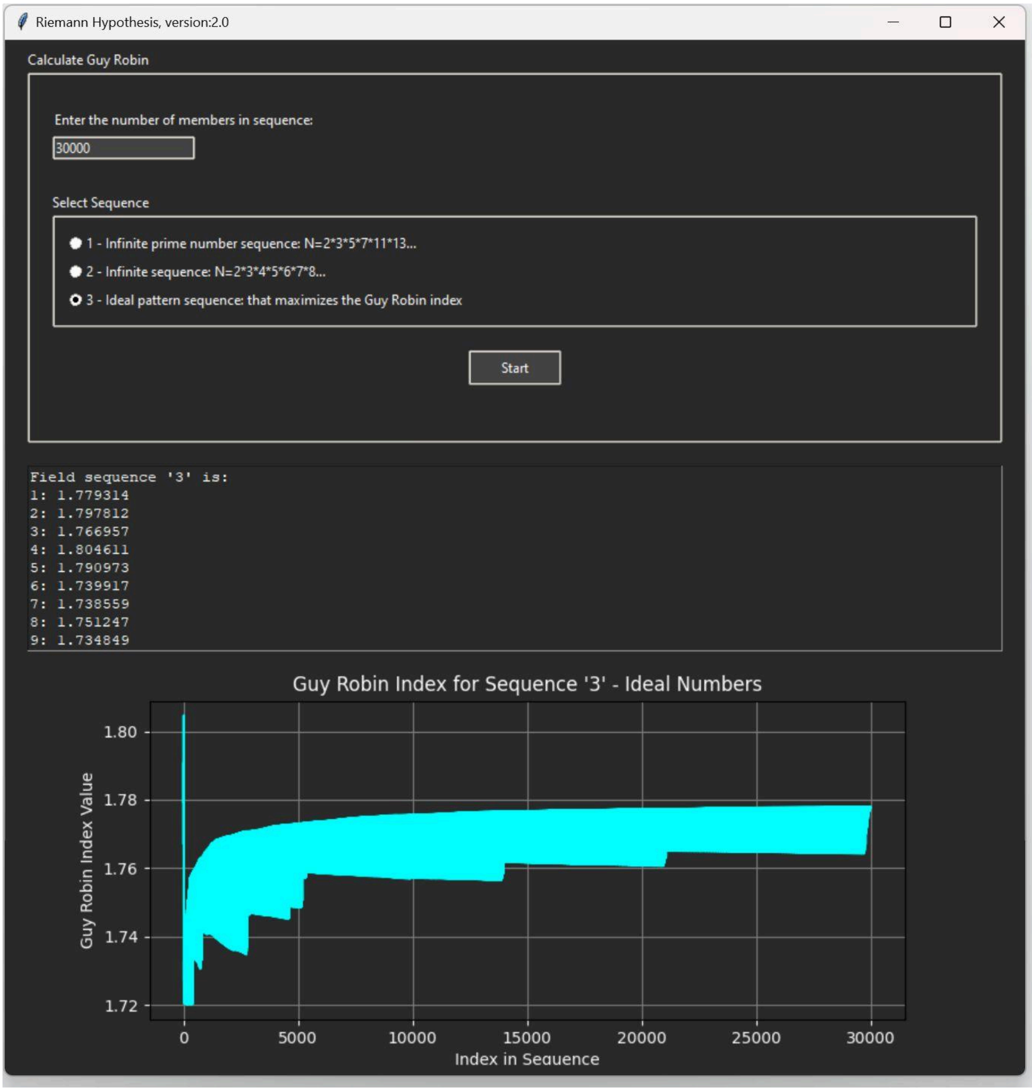
where Hn denotes the nth harmonic number Hn :

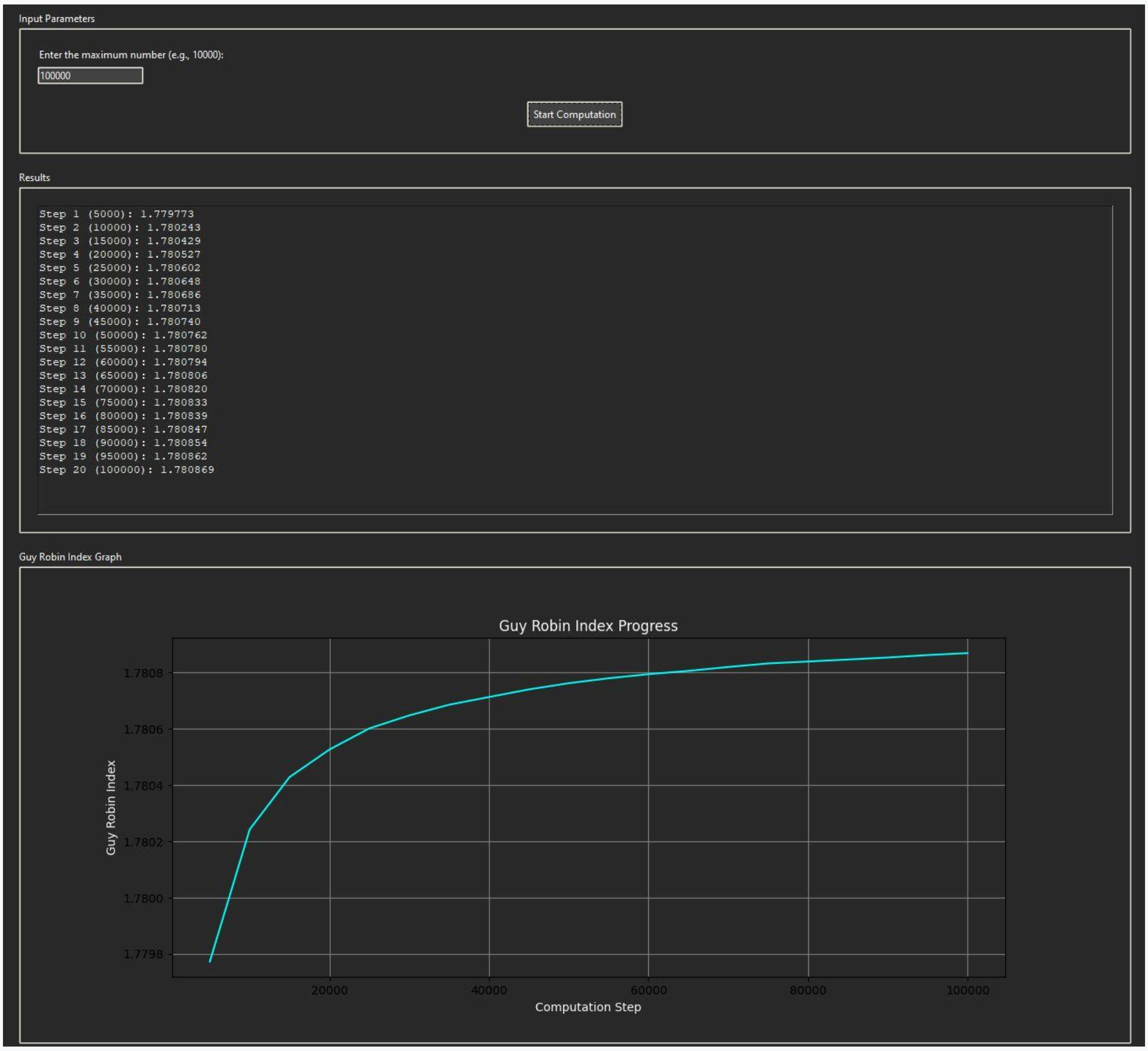
$$Hn = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}...$$

$$\sum_{n=1}^{k} \frac{1}{n} = ln\,k + \gamma + \varepsilon_k \leq ln\,k + 1;\ \varepsilon_k \sim \frac{1}{2\,k}$$

Numerické testy - numerical tests

Numerické testy sú podrobne popísané v článku- referencia (7), a pokročilé testy - referencia (8) / Numerical tests are described in detail in the article - reference (7), and advanced tests - reference (8).





obr.1 Guy Robin test pre menšie a pokročilé testovanie. / Fig.1 Guy Robin test for smaller and advanced testing.

<u>Poznámka:</u> ideálne čísla sú v literatúre nazývané <u>highly composite numbers</u>. / Note: ideal numbers are called highly composite numbers in the literature.

<u>Final verzia Riemann / Final version of Riemann.</u>
referencia (10) / reference (10):
download file in Python - Github : riemanm_hypothesis_final.py

<u>POPIS:</u> Algoritmus obsahuje výpočet Guy Robin rovnice pre rôzne sekvencie, pre preverenie platnosti Riemannovej hypotézy. Optimálne trvá výpočet pre sekvenciu 3: 40000 ideálnych čísel zhruba 1-2 minút.
/ DESCRIPTION: The algorithm contains a calculation of the Guy Robin equation for different sequences, to verify the validity of the Riemann hypothesis. Optimally, the calculation for a sequence of 3: 30000 ideal numbers takes about 1-2 minutes.

Pokročilé testovanie: / Advanced testing
Verzia Riemann test (zdrojový kód programu) / Riemann test version (program source code):
download file in Python - Github: [Riemanm_test.py]

Výsledky / Results: Hodnota e^ γ nebola prekročená. Test bol vykonaný až pre vysoko-zložené číslo pozostávajúce z prvých 500 tisíc prvočísiel násobených medzi sebou. Číslo "N" malo hodnotu až 3201675 číslic.
/ The value of e ^ γ was not exceeded. The test was performed only for highly-composite numbers consisting of the first 500 thousand prime numbers multiplied by each other. The number "N" had a value of up to 3201675 digits.

Výpočet sigma / sigma calculation

Vo vzťahoch pre G(n) v z mysle Gronwall ,Robin vystupuje: / It appears in the relations. for G (n):
$G(n)$:
example:
$n = 12;\ \sigma(n) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

each number can be decomposed, prime decomposition:

$$n = \prod_{i,j} p_i^{j_i};\ n = p_1^{j_1} \cdot p_2^{j_2} \cdot p_2^{j_2} ... p_n^{j_n}\ ; p_i \in prime;\ j_i \in N \qquad (1.1)$$

example:
$n = 2 \cdot 3 \cdot 5 \cdot 7$
$n = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040$
let's define a simple sequence (1):

$$n = \prod_{i}^{n} p_i\ ; n = p_1 \cdot p_2 \cdot p_3 ... p_n\ ; \qquad p_i \in prime \qquad (1.2)$$

then it applies:

$$\sigma(n) = \prod_{p_i \in prime}^{p_n} \left(p_i + 1\right) \qquad (1.3)$$

$$\frac{\sigma(n)}{n} = \prod_{p_i \in prime}^{p_n} \left(\frac{p_i+1}{p_i}\right) = \prod_{p_i \in prime}^{p_n} \left(1 + \frac{1}{p_i}\right) \qquad (1.4)$$

example:
$n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$
$\sigma(n) = (2 + 1) \cdot (3 + 1) \cdot (5 + 1) \cdot (7 + 1) \cdot (11 + 1) \cdot (13 + 1) = 96768$

for (1.1) is $\sigma(n)$:

$$\sigma(n) = \prod_{p_i \in prime}^{p_n} (1 + p_i + p_i^2 + p_i^3 + ... p_i^{j_i}) \quad (1.5)$$

$$\frac{\sigma(n)}{n} = \prod_{p_i \in prime}^{p_n} \frac{(1+p_i+p_i^2+p_i^3+...p_i^{j_i})}{p_i^{j_i}} \quad (1.6)$$

example:

$n = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040$

$\sigma(n) = (1 + 2^1 + 2^2 + 2^3 + 2^4) \cdot (1 + 3^1 + 3^2) \cdot (5 + 1) \cdot (7 + 1) = 19344$

Highly composite numbers

Definujme sekvenciu (3) - vysoko zložené čísla. Vysoko zložené čísla sú také čísla kde maximalizujeme vzťah $\sigma(n)/n$. / let's define a sequence (3) - highly composite numbers: highly composite numbers are numbers that maximize $\sigma(n)/n$.

$$sup \frac{\sigma(n)}{n} = sup \prod_{p_i \in prime}^{p_n} \frac{(1+p_i+p_i^2+p_i^3+...p_i^{j_i})}{p_i^{j_i}} \quad (1.7)$$

example:

$n = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 720720$

$\sigma(n) = 3249792$; $\sigma(n)/n = 4.509..$

let's adjust the equation (1.5):
reference (2), page 9

$$\sigma(n) = \prod_{p_i \in prime}^{p_n} (1 + p_i + p_i^2 + p_i^3 + ... p_i^{j_i}) = \prod_{p_i \in prime}^{p_n} \frac{p_i^{j_i+1}-1}{p_i-1} \quad (1.8)$$

we substitute into the equation (1.7):

$$sup \frac{\sigma(n)}{n} = sup \prod_{p_i \in prime}^{p_n} \frac{p_i^{j_i+1}-1}{(p_i-1)\, p_i^{j_i}} \quad (1.9)$$

$$\frac{\sigma(n)}{n} = (2 - 2^{-j_2})(\frac{3}{2} - \frac{3^{-j_3}}{2})(\frac{5}{4} - \frac{5^{-j_5}}{4})(\frac{7}{6} - \frac{7^{-j_7}}{6})... \quad (1.10)$$

$$\frac{\sigma(n)}{n} = \prod_{p_i \in prime}^{p_n} (\frac{p_i}{p_i-1} - \frac{p_i^{-j_i}}{p_i-1})$$

(2.0)

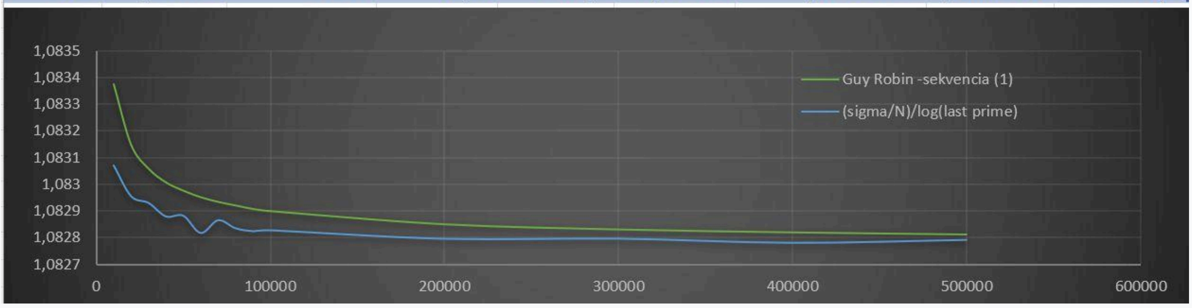
let's define $\beta(n)$:

$$\beta(n) := \prod_{p_i \in prime}^{p_n} \frac{p_i}{p_i - 1} \; ; \quad \beta(n) > sup \frac{\sigma(n)}{n} \tag{2.1}$$

Reformulácia podmienok RH/ Reformulation of RH conditions (Robopol)

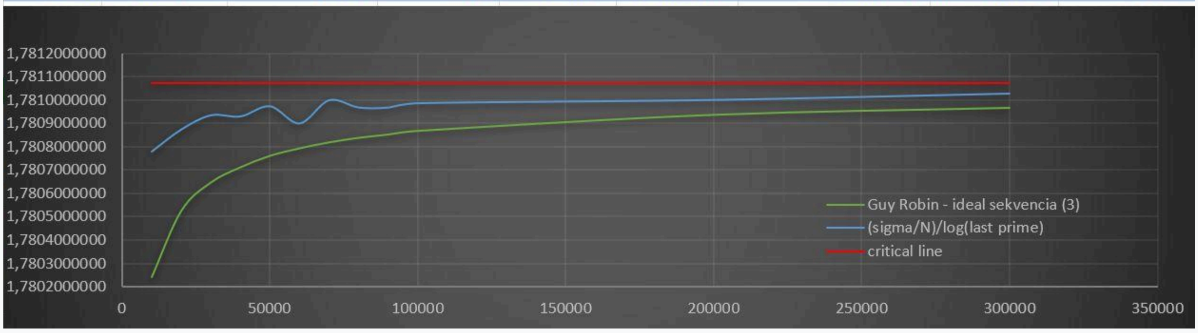
Z empirického testovania s využitím programovacieho jazyka Python je zobrazený priebeh pre sekvenciu (1) a sekvenciu (3). podrobnejšie viď. článok - referencia (8).
/ From empirical testing using the Python programming language, a graph for sequence (1) and sequence (3) is shown. See below. Article link - reference (8).

Numbers	Guy Robin	Last prime	log(last prime)	log(N)	sigma/N	(sigma/N)/log(last prime)	last prime/log(N)	last prime-log(N)
10000	1,083373478	104729	11,559	104392,20	12,519366684	1,0830715834146	1,003226294685	336,800
20000	1,08314743	224737	12,323	224246,33	13,344918374	1,0829553115673	1,002188066217	490,666
30000	1,08305859	350377	12,767	349847,62	13,825516883	1,0829303183978	1,001513173078	529,380
40000	1,083007238	479909	13,081	479167,03	14,165522963	1,0828791396963	1,001548458433	741,970
50000	1,082976324	611953	13,324	611242,25	14,428762833	1,0828818693551	1,001162801119	710,753
60000	1,082951903	746773	13,524	745512,21	14,643488059	1,0828165898938	1,001691172681	1260,790
70000	1,082934562	882377	13,690	881602,55	14,824829020	1,0828651052147	1,000878457612	774,450
80000	1,082921205	1020379	13,836	1019250,15	14,981757627	1,0828345663934	1,001107527661	1128,848
90000	1,082908032	1159523	13,964	1158260,16	15,120027131	1,0828235230164	1,001090289259	1262,839
100000	1,082899989	1299709	14,078	1298474,43	15,243658949	1,0828268864969	1,000950783036	1234,567
200000	1,082851067	2750161	14,827	2748048,88	16,054784923	1,0827949573727	1,000768587455	2112,116
300000	1,082831675	4256233	15,264	4254062,07	16,527676597	1,0827954815872	1,000510318846	2170,928
400000	1,082820715	5800079	15,573	5796662,79	16,862542769	1,0827797497634	1,000589340806	3416,210
500000	1,082812857	7368787	15,813	7366416,45	17,121915404	1,0827908243885	1,000321805122	2370,551



tab. no.1 sequence (1) for very high numbers.

Numbers	Guy Robin	Last prime	log(last prime)	log(N)	sigma/N	(sigma/N)/log(last prime)	last prime/log(N)	last prime-log(N)
10000	1,7802427491	104729	11,559	105093,26	20,584240862	1,7807774871308	0,996533962471	-364,257
20000	1,7805266280	224737	12,323	225275,23	21,945129891	1,7808722625548	0,997610787790	-538,230
30000	1,7806479515	350377	12,767	351094,75	22,736757893	1,7809333765146	0,997955672806	-717,753
40000	1,7807132180	479909	13,081	480668,24	23,296950973	1,7809284057179	0,998420452129	-759,238
50000	1,7807623951	611953	13,324	612914,28	23,730404715	1,7809721675853	0,998431625874	-961,279
60000	1,7807944773	746773	13,524	747363,69	24,084011587	1,7808985942047	0,999209639516	-590,687
70000	1,7808200644	882377	13,690	883585,73	24,382531727	1,7809981315336	0,998632014147	-1208,733
80000	1,7808394634	1020379	13,836	1021389,00	24,640895149	1,7809668050790	0,999011147260	-1010,003
90000	1,7808536348	1159523	13,964	1160551,80	24,868563435	1,7809667428002	0,999113523359	-1028,802
100000	1,7808693236	1299709	14,078	1300902,18	25,072090884	1,7809854052338	0,999082803892	-1193,182
200000	1,7809379736	2750161	14,827	2751565,60	26,407179475	1,7809993040522	0,999489525411	-1404,604
300000	1,7809678802	4256233	15,264	4258379,16	27,185404626	1,7810266991942	0,999496015620	-2146,157



tab. no.2 sequence(3) for very high numbers.

for sequence (1) tab.no.1:

$$initial\ test:\ n = \prod_{p_1 \in prime}^{p_{10000}} p_i;\ last\_prime = p_{1000} = 104729$$

$$end\ test: n = \prod_{p_1 \in prime}^{p_{500\,000}} p_i;\ last\_prime = p_{500\,000} = 7368787$$

for sequence (3) - highly composite numbers, tab.no.2:
in terms of the equation (1.7) for highly composite numbers

$$initial\ test:\ n = sup \prod_{p_1 \in prime}^{p_{10000}} p_i^{j_i};\ last\_prime = p_{1000} = 104729$$

$$end\ test: n = sup \prod_{p_1 \in prime}^{p_{300\,000}} p_i^{j_i};\ last\_prime = p_{300\,000} = 4256233$$

Numerical testing shows that the following statement holds:

$$last\_prime = p_n;\quad log(n) \approx p_n \tag{2.2}$$

for sequence (1):

$$log\ (n) < last\_prime;\quad or\quad log(n) < p_n \tag{2.3}$$

for sequence (3) - highly composite numbers:

$$log\ (n) > last\_prime;\quad or\quad log(n) > p_n \tag{2.4}$$

Lagarias theorem:

$$\sigma(n) < log\ (n) + \gamma + \varepsilon + e^{ln(n)+\gamma+\varepsilon}\ log(log\ (n) + \gamma + \varepsilon) =$$

$$= log\ (n) + \gamma + \varepsilon + n\, e^{\gamma} e^{\varepsilon} \, log(log\ (N) + \gamma + \varepsilon) \tag{2.5}$$

$$\frac{\sigma(n)}{n} < \frac{log\ (n)+\gamma+\varepsilon+n\cdot e^{\gamma}\cdot e^{\varepsilon}\cdot log(log(n)+\gamma+\varepsilon)}{n}$$
(2.6)

creates a limit for Gronwall theorem, while for $N \rightarrow \infty, \varepsilon = 0$

$$\lim_{n\rightarrow\infty} \frac{\sigma(n)}{n\ log\ (log(\ n))} = \lim_{n\rightarrow\infty} \frac{log(n)+\gamma+n\ e^{\gamma}\ log(log(n)+\gamma)}{n\ log\ (log(n))} = e^{\gamma}$$
(2.7)

Guy Robin theorem :

$$\frac{\sigma(n)}{n} < e^{\gamma} log\ (log\ n)\ ;\ \ n > 5040 \tag{2.8}$$

Guy Robin theorem is a stronger statement because:

$$e^{\gamma} log\ (log\ n) < \frac{log\ (n)+\gamma+\varepsilon+n\ e^{\gamma} e^{\varepsilon} ln(ln\ (n)+\gamma+\varepsilon)}{n}\ log\ (log\ (n)) \tag{2.9}$$

V tejto stati vznikne silnejší teorém a pokiaľ sa ho podarí preukázať, platia aj zvyšné Robin, Lagarias theorem. Tento teorém je naviazaný na vysoko - zložené čísla, kde každé číslo "n" je práve vysoko - zložené číslo.
/eng: A stronger theorem arises in this article, and if it can be proved, the remaining Robin, Lagarias theorem, also applies. This theorem is related to highly-composite numbers, where each number "n" is just a highly-composite number.

<u>Robopol theorem:</u>
for highly composite numbers in terms of equation (1.7), (2.1) we get:

$$\beta(n) < e^{\gamma} log\ (log\ n) \tag{3.0}$$

$$\prod_{p_i \in prime}^{p_n} \frac{p_i}{p_i - 1} < e^{\gamma} log\ (log(n)); \tag{3.1}$$

for highly composite numbers - n; $if\ p_n \geq p_{10}$
for highly composite numbers in terms of equation (2.4), tab. no.2 we get:

$$\prod_{p_i \in prime}^{p_n} \frac{p_i}{p_i - 1} < e^{\gamma} log(p_n) \tag{3.2}$$

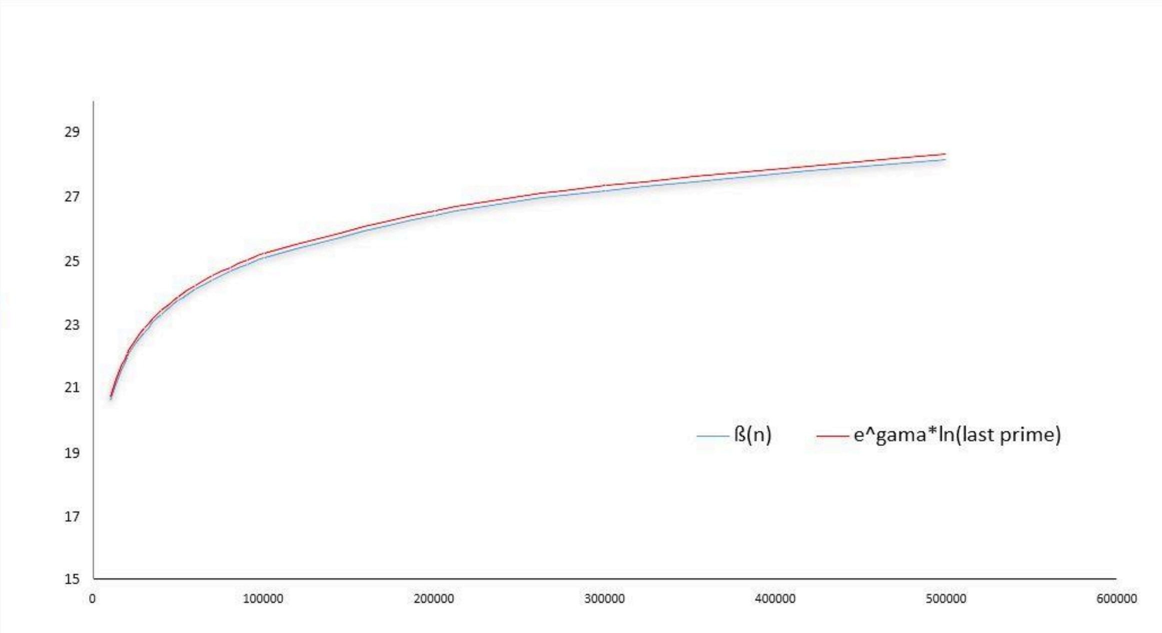
for highly composite numbers - n; $if\ p_n \geq p_{100}$

Equation (3.1) and (3.2) is a stronger statement than equation (2.8) because:
$\beta(n) > sup\ \frac{\sigma(n)}{n}$ and $log(n) > p_n\ - for\ highly\ composite\ number\ "n"$

Test Robopol theorem

Program (in Python) test Robopol theorem:
reference (10)
Download file in GitHub: sigma-max_test.py
The table no.3 shows the test results for very large numbers:

Numbers	Last prime	ln(last prime)	ß(n)	e^gama*ln(last prime)	
10000	104729	11,55913	20,59352	20,7021	0,10858
20000	224737	12,32269	21,9515	22,0696	0,1181
30000	350377	12,76676	22,74206	22,86494	0,12288
40000	479909	13,08135	23,30135	23,42835	0,127
50000	611953	13,32441	23,73436	23,86366	0,1293
60000	746773	13,52352	24,08757	24,22026	0,13269
70000	882377	13,69037	24,38586	24,5191	0,13323
80000	1020379	13,83568	24,644	24,77934	0,13534
90000	1159523	13,96352	24,87145	25,00829	0,13684
100000	1299709	14,07765	25,07481	25,2127	0,13789
200000	2750161	14,82717	26,40906	26,55507	0,146
300000	4256233	15,2639	27,18694	27,33723	0,15029
500000	7368787	15,81276	28,16442	28,32024	0,15582
1000000	15485863	16,55544	29,48665	29,65035	0,1637
2000000	32452843	17,2953	30,80453	30,97542	0,17089



tab. No. 3 Test robopol theorem for very large numbers.

Z tabuľky a grafu je vidieť, že je teorém veľmi tesne splnený a smerom k nekonečnu sa mierne priebehy rozchádzajú. To znamená, že teorém by mohol platiť (ako silnejšie tvrdenie oproti Robin teorému) do nekonečna. Na to však potrebujeme dôkaz. Samozrejme to platí v zmysle rovnice (3.2) pre vysoko- zložené čísla, ktoré sú už dostatočné veľké.
/eng: it can be seen from the table and the graph that the theorem is very tightly fulfilled and they differ slightly towards infinity. This means that the theorem could apply (as a stronger

statement compared to Robin's theorem) to infinity. But we need proof of that. Of course, in the sense of equation (3.2) for highly composite numbers that are already large.

Approximation pi(x)

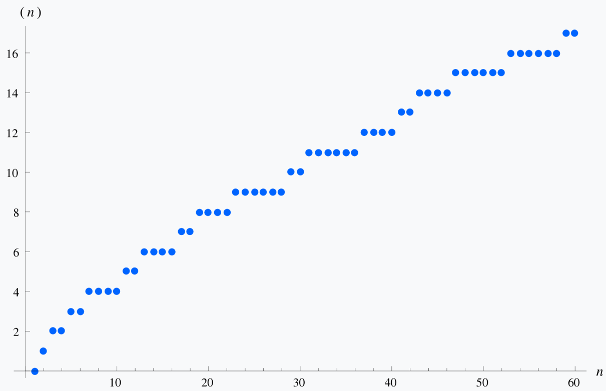


Figure no.2 The values of $\pi(n)$ for the first 60 positive integers,source: wiki

Prime- counting function:

$$\pi(x) \approx \frac{x}{log(x)}; \quad year: 1792 \tag{3.3}$$

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{log(x)}} = 1 \tag{3.4}$$

better approximation π(x):

$$\pi(x) \approx Li(x) := \int_2^x \frac{dt}{log\,(t)} \tag{3.5}$$

$$\lim_{x \to \infty} Li(x)/\pi(x) = 1 \tag{3.6}$$

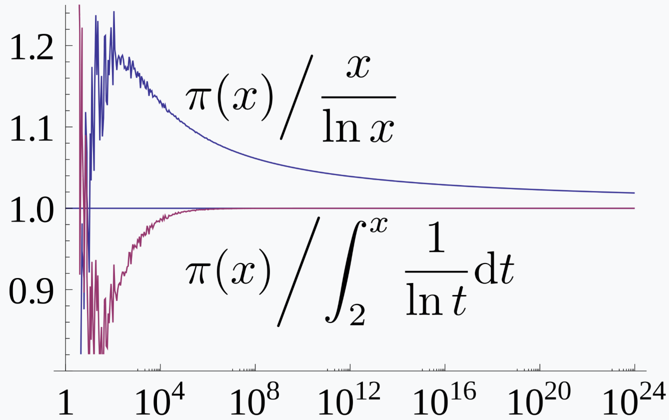


Figure no.3 Graph showing ratio of the prime-counting function π(x) to two of its approximations, x/log x and Li(x).Source: wiki.

Podľa zdroja (3) platí: /According to source (3):
to read:

In Pierre Dusart's thesis there are stronger versions of this type of inequality that are valid for larger x. Later in 2010, Dusart proved:

$$\frac{x}{log\,(x)-1} < \pi(x) < \frac{x}{log\,(x)-1.1} \quad (3.7)$$

The proof by de la Vallée Poussin implies the following: For every ε > 0, there is an S such that for all x > S,

$$\frac{x}{log\,(x)-(1-\varepsilon)} < \pi(x) < \frac{x}{log\,(x)-(1+\varepsilon)} \quad (3.8)$$

Vzťah (3.8) je veľmi dôležitý pre následné dokazovanie. Hovorí, že aproximácia $\frac{x}{log\,(x)-1}$ je tá, ktorá by mala byť zhruba rovná $\pi(x)$smerom k nekonečnu. Teda, že ide o nejakú kritickú hranicu (stred), linku. Zapíšme to:

$$\pi(x) \approx \frac{x}{log\,(x)-1} \quad (3.9)$$

$$\lim_{x\to\infty} \pi(x) = \lim_{x\to\infty} \frac{x}{log\,(x)-1} \quad (3.10)$$

ENG: ***Equation (3.8) is very important for subsequent evidence.*** That means that the approximation $\frac{x}{log\,(x)-1}$ is the one that should be roughly equal to $\pi(x)$ towards infinity. That is, it is a critical boundary, a line.

example:

$x = 10^{25}$

$\pi(10^{25}) = 176846309399143769411680$

$\frac{10^{25}}{log\,(10^{25})-(1+0.019)} - \pi(10^{20}) \sim 2.02482 \cdot 10^{18}$

$\frac{10^{25}}{log\,(10^{25})-(1+0.019)} > \pi(10^{25})$

Robopol teorém- súvislosti /context in Robopol's theorem

V zmysle rovnice (3.2) odvodíme ďalšie súvislosti, ktoré by mali platiť. /According to equation (3.2), we derive other contexts that should apply.

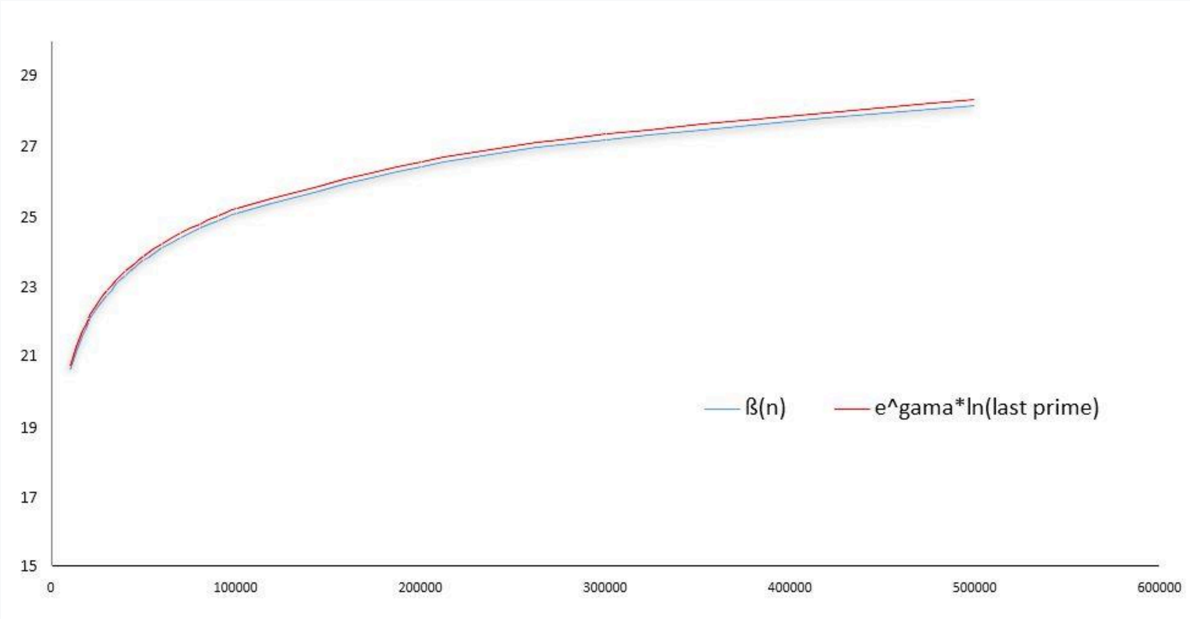


Figure no.4 : Development chart for robopol theorem (for very large numbers).

Tvrdenie (1.0)/Statement (1.0):
Nech platí na celom definičnom obore rovnice (3.2) / Let equation (4.0) hold for the domain of the equation (3.2):

for $p_k \geq p_{100}$; $\beta(p_{k+1}) - \beta(p_k) \approx e^{\gamma} log(p_{k+1}) - e^{\gamma} log(p_k)$ (4.0)

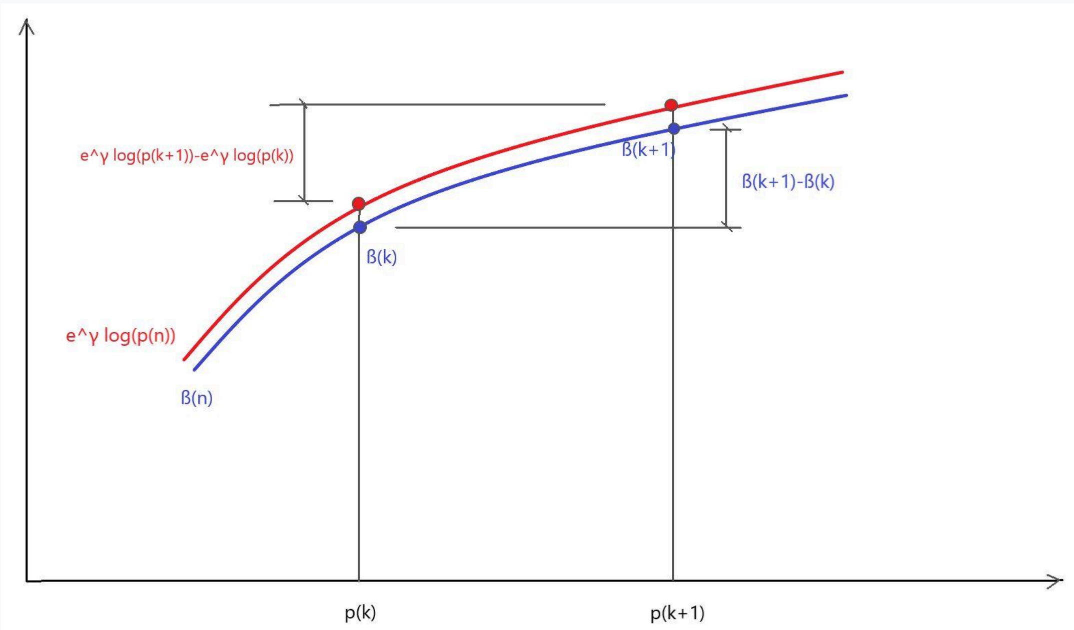


Figure no.5 expression of equation 4.0.

Na obr. č.5 je vyjadrenie tvrdenia (1.0) v geometrickej interpretácií. /In FIG. No. 5 is the expression of statement (1.0) in geometric interpretation.

modify equation (4.0):

$$\prod_{p_i \in prime}^{p_{k+1}} \left(\frac{p_i}{p_i - 1}\right) - \prod_{p_i \in prime}^{p_k} \left(\frac{p_i}{p_i - 1}\right) \approx e^{\gamma} \, log(p_{k+1}) - e^{\gamma} \, log(p_k) \quad (4.1)$$

$$\varepsilon_k = \prod_{p_i \in prime}^{p_k} \left(\frac{p_i}{p_i - 1}\right); \ \omega_k = log\,(p_k)\, e^{\gamma} \quad (4.2)$$

V zmysle obr. č.6 stotožníme počiatočné body v p_k

/ identity first point in p_k: $\varepsilon_k = \omega_k$

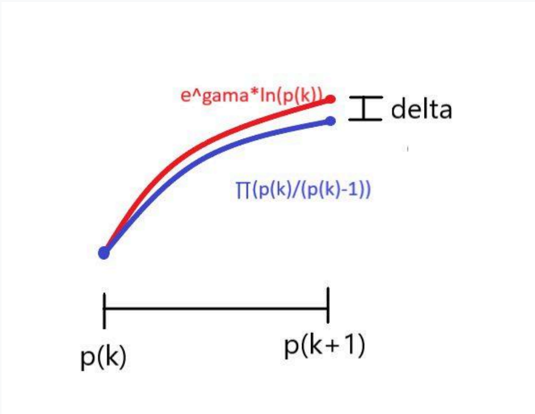


Figure no.6 identification of starting points in p_k, $\varepsilon_k = \omega_k$.

$$\Delta\varepsilon_k = \varepsilon_k \frac{p_{k+1}}{p_{k+1} - 1} - \varepsilon_k = \varepsilon_k \left(\frac{p_{k+1}}{p_{k+1} - 1} - 1\right) \quad (4.3)$$

$$\Delta\omega_k = e^{\gamma}\,(log(p_{k+1}) - log(p_k)) \quad (4.4)$$

in terms of the equation (4.1):

$$\Delta\omega_k \approx \Delta\varepsilon_k; \quad e^{\gamma}\,(ln(p_{k+1}) - ln\,(p_k)) \approx \varepsilon_k \frac{p_{k+1}}{p_{k+1} - 1} - \varepsilon_k \quad (4.5)$$

substitute into equation (4.5) $\varepsilon_k = \omega_k$, we get:

$$e^{\gamma}\, log(p_{k+1}) - e^{\gamma}\, log(p_k) \approx log(p_k)\, e^{\gamma} \frac{p_{k+1}}{p_{k+1} - 1} - log(p_k)\, e^{\gamma} \quad (4.6)$$

$$log(p_{k+1}) \approx log(p_k) \frac{p_{k+1}}{p_{k+1} - 1} \quad (4.7)$$

<u>example:</u>

$p_{999\,999\,999} = 22801763477$

$p_{1\,000\,000\,000} = 22801763489$

$log(22801763489) \approx log\,(22801763477) \cdot \frac{22801763489}{22801763489 - 1}$

$23.850103715924.. \approx 23.850103715442...$

Tvrdenie (2.0)/Statement (2.0):
Nech existuje hladká, spojitá funkcia $g(x) := f(x)$, ktorá aproximuje $\pi(x)$ tak, že platí smerom k nekonečnu nasledovný vzťah:
/Let there exist a smooth, continuous function g (x): = f (x), which approximates (x) such that the following relation holds towards infinity:

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \pi(x) \qquad (4.8)$$

Tvrdeniu (2.0) vyhovuje v zmysle rovnice (3.8) a (3.9):
/Theorem (2.0) is satisfied in accordance with equations (3.8) and (3.9):

$$g(x) = \frac{x}{log\,(x) - 1} \qquad (4.9)$$

Tvrdenie (3.0)/statement (3.0):
Nech platí na definičnom obore funkcie $g(x)$ pre všetky $x \geq 100$ v zmysle rovnice (4.7) táto rovnica:
/Let the following equation hold on the domain of function g (x) for all x100 in the sense of equation (4.7):

$$log(x + \Delta x) \geq log(x)\,\frac{x + \Delta x}{x + \Delta x - 1} \qquad (4.10)$$

kde/where
Δx - is the horizontal distance in the sense figure no.7.

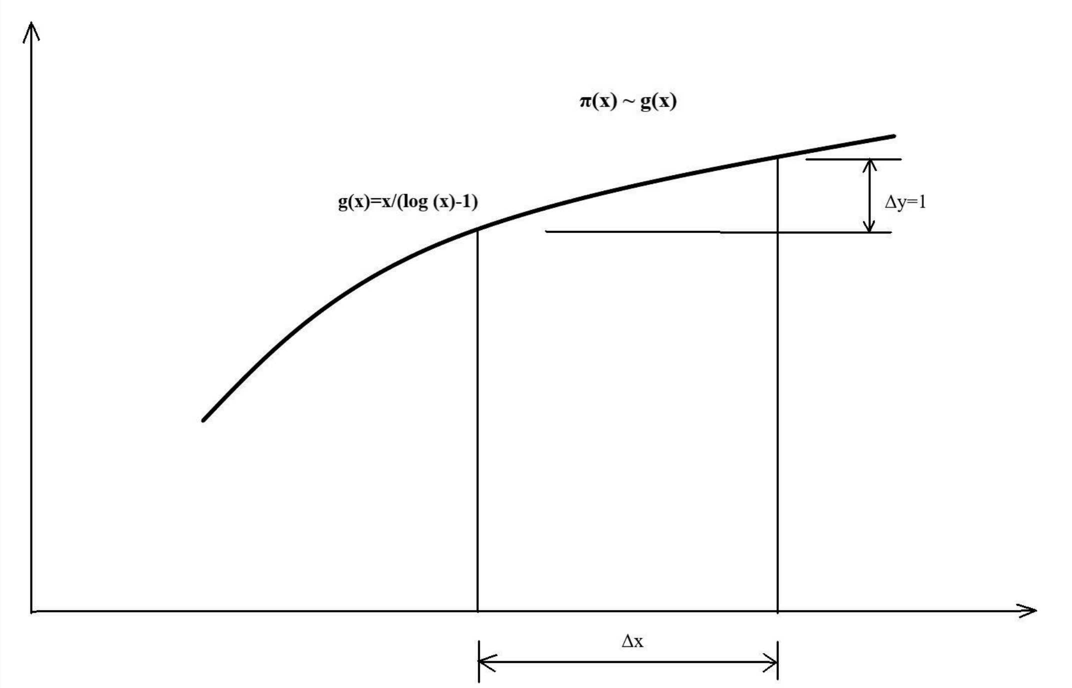


Figure no.7 Horizontal distance Δx in the equation (4.10)

Zadefinujeme najskôr rovnicu / let's define the equation in the sense of the figure no.7:

$$\frac{x}{log(x)-1} + 1 = \frac{x+\Delta x}{log(x+\Delta x)-1} \quad (5.0)$$

or

$$\frac{x}{log(x)-1} = 1 + \frac{x-\Delta x}{log(x-\Delta x)-1} \quad (5.1)$$

Pointa tvrdení (1.0), (2.0) a (3.0) je v tom, že ak je splnená rovnica (4.10) na celom definičnom obore $x > 100$, pre všetky Δx, ktoré vypočítame z rovnice (5.1), potom nutne platí aj teorém - rovnica (3.0), (3.2). Zadefinujeme to do tvrdenia (4.0).
/The point of the statements (1.0), (2.0) and (3.0) is that if equation (4.10) is satisfied on the whole domain x> 100, for all x, which we calculate from equation (5.1), then the theorem - equation (3.0), (3.2). We will define this in the statement (4.0).

Tvrdenie (4.0)/ statement (4.0):
Ak platí rovnica (4.10) pre všetky $x > 100$, pričom Δx vypočítame z rovnice (5.1), potom nutne platí aj teorém - rovnica (3.0), (3.2).
Eng: If equation (4.10) applies to all $x > 100$, where Δx is calculated from equation (5.1), then equation (3.0), (3.2) necessarily also applies.

Vysvetlenie:
Rovnica (4.10) je pomerne jednoduchá a intuitívna.Táto rovnica je už spojitá oproti pôvodnej (4.7). Pôvodná rovnica (4.7) obsahovala v sebe približne. No to len z toho dôvodu, že prvočísla v $\pi(x)$ obsahujú drobné fluktuácie od nejakej strednej hodnoty vyhladenej funkcie, ktorá by ju dokázala dokonale aproximovať (ako hladká krivka, bez ozubenia). Z numerických testov (tab. no.3) sa práve ukázalo, že vo väčšine prípadov bolo splnené:

$$log(p_{k+1}) > log(p_k)\frac{p_{k+1}}{p_{k+1}-1} \quad (5.2)$$

Teda, aby sme potvrdili platnosť rovnice (3.0), (3.2), tak preskúmame platnosť (5.2) v ľubovolnom bode p_k a jeho suseda p_{k+1} s tým rozdielom, že to robíme na vyhladenej funkcii $g(x)$ pre ľubovoľné $x > 100$. No ak preukážeme platnosť pre všetky $x \to \infty$, potom je zrejmé, že aj rovnica (3.0), (3.2) musí platiť.
Niekto by mohol namietať, že predsa $g(x)$ neaproximuje $\pi(x)$ dobre. Teda pre malé x platí, že $g(x) < \pi(x)$. No zároveň vieme, že limitne $g(x)$ doháňa $\pi(x)$, v zmysle rovnice (4.8). Teda ak má limitne $g(x)$ dohnať $\pi(x)$, tak je zjavné, že smerom k nekonečnu -sklon $g(x)$ rastie viac ako $\pi(x)$.
Z numerických testov tab. no. 3 je však preverená rovnica (3.0) (3.2) do veľkých čísiel. Presnejšie povedané sú preverené vysoko - zložené čísla pre $p_n \geq p_{100}$ po tie uvedené v

tab.no.3. Zároveň vidíme, že krivky sa mierne rozchádzajú a teda má stále viac navrch $e^{\gamma}log\ (p_n)$.
V nasledujúcich statiach sa teda pokúsim dokázať, že kľúčové tvrdenie (4.0) naozaj platí.

/eng:
<u>Explanation:</u>
Equation (4.10) is relatively simple and intuitive. This equation is already continuous compared to the original (4.7). The original equation (4.7) contained approximately. But that's just because the primes in $\pi(x)$ contain slight fluctuations from some mean value of the smoothed function that could approximate it perfectly (like a smooth curve, no toothing). Numerical tests (Table no.3) have just shown that in most cases the following were met:

$$log(p_{k+1}) > log(p_k) \frac{p_{k+1}}{p_{k+1}-1} \qquad (5.2)$$

Thus, to confirm the validity of equation (3.0),(3.2), we examine the validity (5.2) at any point p_k and its neighbor p_{k+1}, with the difference that we do this on the smoothed function $g(x)$ for any $x > 100$. But, if we prove the validity for all $x \to \infty$, then it is obvious that equation (3.0) must also hold.
One might argue that $g(x)$ does not approximate $\pi(x)$ well. Thus, for small x, $g(x) < \pi(x)$. But, at the same time we know that the limit $g(x)$ catches up with $\pi(x)$, in the sense of equation (4.8). Thus, if the limit $g(x)$ is to catch up with $\pi(x)$, then it is obvious that towards the infinite - slope $g(x)$ grows more than $\pi(x)$.
From the numerical tests tab. no. 3, however, equation (3.0), (3.2) is tested in large numbers. More precisely, the high-composite numbers for $p_n \geq p_{100}$ are verified after those listed in tab.no.3.In the following sections, I will therefore try to prove that the key statement (4.0) is indeed valid.

Upravme rovnicu (4.10) nasledovne / Adjust equation (4.10) as follows:

$$log(x + \Delta x) \geq log(x) \frac{x+\Delta x}{x+\Delta x-1}$$

let's make a substitution:

$$x + \Delta x = t$$

$$log(t) \geq log(t - \Delta x) \frac{t}{t-1}$$

$$log(t - \Delta x) \geq \frac{log(t)\cdot(t-1)}{t}$$

eliminate logarithm:

$$t - \Delta x \geq t^{(t-1)/t}$$

$$\Delta x \geq t - t^{(t-1)/t} \qquad (5.3)$$

define Δx_{min}:

$$\Delta x_{min} := t - t^{(t-1)/t} \tag{5.4}$$
$$\Delta x \geq \Delta x_{min} \tag{5.5}$$

derivácia funkcie / derivation of a function:
$$\frac{d}{dt}(t - t^{(t-1)/t}) = t^{-(t+1)/t}(t + log(t) - 1) \tag{5.6}$$
limita funkcie / limit function
$$\lim_{t \to \infty} t^{-(t+1)/t}(t + log(t) - 1) = 1 \tag{5.7}$$

example:

$t = 1000,\ \Delta x_{min} = 1000 - 1000^{(1000-1)/1000} = 6.883951579..$

$x = t - \Delta x_{min} = 1000 - 6.883951579 = 993.116048420..$

$log(x + \Delta x_{min}) = log(x) \cdot \frac{x+\Delta x_{min}}{x+\Delta x_{min}-1}$

$log(1000) = log(993.1160484) \cdot \frac{993.1160484+6.883951579}{993.1160484+6.883951579-1}$

6.907755279...=6.907755279...

Analýza aproximácie / Approximation analysis $\pi(x) = \frac{x}{log\ (x)}$

Najskôr preveríme jednoduchší prípad aproximácie $\pi(x)$. / We first examine the simpler case of approximation $\pi(x)$.

In the sense of picture no.7 (for approximation $\pi(x) = \frac{x}{log\ (x)}$) we get the equation:
$$\frac{x}{log\ x} = 1 + \frac{x-\Delta x}{log\ (x-\Delta x)} \tag{6.0}$$

The real analytical solution to the equation is
$$\Delta x = x - e^{-W_{-1}(log(x)/(log(x)-x))}; \quad for\ \Delta x > 0 \tag{6.1}$$
or
$$\Delta x = x - \frac{log(x)-x}{log\ x} W_{-1}(\frac{log(x)}{log(x)-1}); \quad for\ \Delta x > 0 \tag{6.2}$$

$W_{-1}(z)$ $-$ $is\ Lambert\ function,\ reference\ (4)$

derivácia funkcie / derivation of a function:
$$\frac{d}{dx}(\frac{log(x)-x}{log\ x} W_{-1}(\frac{log(x)}{log(x)-1})) =- \frac{(log\ (x)-1)\ W_{-1}(\frac{log(x)}{log(x)-1})^2}{log^2(x)\ (W_{-1}(\frac{log(x)}{log(x)-1})+1)} \tag{6.3}$$
limita funkcie / limit function:

$$\lim_{x \to \infty} - \frac{(log\ (x)-1)\ W_{-1}(\frac{log(x)}{log(x)-1})^2}{log^2(x)\ (\ W_{-1}(\frac{log(x)}{log(x)-1})+1)}$$

answer:
Z hľadiska štandardných matematických funkcií sa nenašiel žiadny výsledok. /No result found in terms of standard mathematical functions.

V zmysle rovnice (5.5) pre $x > 100$ musí platiť / According to equation (5.5), for $x > 100$, it must hold:

$$\Delta x \geq \Delta x_{min};\quad x - \frac{log(x)-x}{log(x)} \cdot W_{-1}(\frac{log(x)}{log(x)-x}) \geq x - x^{(x-1)/x}$$

(6.4)

example(1):

$x = 10^6$

$\Delta x = \frac{log(10^6)-10^6}{log(10^6)} \cdot W_{-1}(\frac{log(10^6)}{log(10^6)-10^6}) = 14.89353360214...$

$\Delta x_{min} = 10^6 - (10^6)^{(10^6-1)/10^6} = 13.815415124237...$

$\Delta x > \Delta x_{min};\ \Delta x - \Delta x_{min} = 1.078118477907...$

example(2):

$x = 10^{200}$

$\Delta x = \frac{log(10^{200})-10^{200}}{log(10^{200})} \cdot W_{-1}(\frac{log(10^{200})}{log(10^{200})-10^{200}}) = 461.519194796772488...$

$\Delta x_{min} = 10^{200} - (10^{200})^{(10^{200}-1)/10^{200}} = 460.517018598809...$

$\Delta x > \Delta x_{min};\ \Delta x - \Delta x_{min} = 1.00217...$

Z numerických výpočtov by sme očakávali, že limita / From numerical calculations, we would expect the limits to:

$$\lim_{x \to \infty} \Delta x - \Delta x_{min} = 1$$

Upravme rovnicu (6.0) na tento požadovaný tvar / Adjust equation (6.0) to this desired shape:

$$\frac{x}{log\ x} - 1 - \frac{x-\Delta x}{log\ (x-\Delta x)} = 0 \qquad (6.5)$$

next:

$$\frac{x\ log(x-\Delta x) - log(x)\ log(x-\Delta x) - log(x)\ (x-\Delta x)}{log(x)\ log(x-\Delta x)} = 0$$

next:

$$(x - log(x))\, log(x - \Delta x) - log(x)(x - \Delta x) = 0 \quad (6.6)$$

Urobme Taylorov rozvoj podľa x dostaneme: / Let's expansion series at $x = \infty$:

$$(\Delta x + \Delta x\, log(x) - log^2(x)) - \frac{\Delta x\,(\Delta x - 2\, log(x))}{2\, x} + \frac{\Delta x^2\,(3\, log(x) - 2\,\Delta x)}{6\, x^2} + \frac{\Delta x^3\,(4\, log(x) - 3\,\Delta x)}{12\, x^3} + \frac{\Delta x^4\,(5\, log(x) - 4\,\Delta x)}{20\, x^4} + \frac{\Delta x^5\,(6\, log(x) - 5\,\Delta x)}{30\, x^5} + \frac{\Delta x^6\,(7\, log(x) - 6\,\Delta x)}{42} + O\left(\left(\frac{1}{x}\right)^7\right)$$

(6.7)

$for\ x \rightarrow \infty\ the\ equation\ is\ reduced$:

$$\Delta x + \Delta x\, log(x) - log^2(x) = 0 \quad (6.8)$$

vyjadríme z rovnice Δx: / we get from the equation Δx:

$$\Delta x = \frac{log^2(x)}{log(x) - 1} \quad (6.9)$$

Podľa (6.4) vypočítame limitu / According to (6.4) we calculate the limit:

$$\lim_{x \to \infty} \Delta x - \Delta x_{min} = \lim_{x \to \infty} \frac{log^2(x)}{log(x) - 1} - \left(x - x^{(x-1)/x}\right) = 1 \quad (6.10)$$

Result:

Rovnica (6.10) ukázala, že rovnica $\Delta x \geq \Delta x_{min}$ platí až do nekonečna. / Equation (6.10) showed that equation $\Delta x \geq \Delta x_{min}$ is true to infinity.

Analýza aproximácie / Approximation analysis $\pi(x) = \frac{x}{log\,(x) - 1}$

Teraz preveríme aproximáciu $\pi(x) = \frac{x}{log\,(x) - 1}$ v zmysle rovnice (3.9), (3.10). / We now check the approximation π(x) = x/(log (x) -1) in the sense of equation (3.9), (3.10).

Riešením rovnice (5.1) je / The solution of equation (5.1) is:

$$\Delta x = \frac{(x - log(x) + 1) \cdot W_{-1}\left(-\frac{\sqrt[1+x-log(x)]{\frac{e^{x+1}}{x}} \cdot (log\,(x) - 1)}{x - log\,(x) + 1}\right)}{log\,(x) - 1} + x \quad (7.0)$$

$W_{-1}(z)\ -\ is\ Lambert\ function,\ reference\ (4)$

V zmysle rovnice (5.5) pre $x > 100$ musí platiť / According to equation (5.5), for $x > 100$, it must hold:

$\Delta x \geq \Delta x_{min}$;

$$\frac{(x-log(x)+1)\cdot W_{-1}\left(-\frac{\sqrt[1+x-log(x)]{\frac{e^{x+1}}{x}}\cdot(log\,(x)-1)}{x-log\,(x)+1}\right)}{log\,(x)-1} + x \geq x - x^{(x-1)/x} \qquad (7.1)$$

example:

$x = 10^{180}$

$$\Delta x = \frac{(10^{180}-log(10^{180})+1)\cdot W_{-1}\left(-\frac{\sqrt[1+10^{180}-log(10^{180})]{\frac{e^{10^{180}+1}}{10^{180}}}\cdot(log\,(10^{180})-1)}{10^{180}-log\,(10^{180})+1}\right)}{log\,(10^{180})-1} + 10^{180} =$$

$= 414.467741185201..$

$\Delta x_{min} = 10^{180} - (10^{180})^{(10^{180}-1)/10^{180}} = 414.465316...$

$\Delta x > \Delta x_{min}$; $\Delta x - \Delta x_{min} \sim 0$

Z numerických výpočtov by sme očakávali, že limita / From numerical calculations, we would expect the limits to:

$$\lim_{x\to\infty} \Delta x - \Delta x_{min} = 0$$

Upravme rovnicu (5.1) na tento požadovaný tvar / Adjust equation (5.1) to this desired shape:

$$\frac{x}{log\,x-1} - 1 - \frac{x-\Delta x}{log\,(x-\Delta x)-1} = 0 \qquad (7.2)$$

next:

$$\frac{x}{log(x)-1} - \frac{x}{log(x-\Delta x)-1} + \frac{\Delta x}{log(x-\Delta x)-1} - 1 = 0 \qquad (7.3)$$

Urobme rozvoj podľa x dostaneme: /Let's expansion series at $x = \infty$:

Puiseux series

$$-\frac{2\,\Delta x-(\Delta x-2)\,log(x)+log^2(x)+1}{(log(x)-1)^2} + \frac{\Delta x^2(log(x)-3)}{2\,x\,(log(x)-1)^3} + O\left(\left(\frac{1}{x}\right)^2\right)$$

(7.4)

$for\ x \to \infty\ the\ equation\ is\ reduced:$

$$-\frac{2\,\Delta x-(\Delta x-2)\,log(x)+log^{2}(x)+1}{(log(x)-1)^{2}} = 0 \qquad (7.5)$$

vyjadríme z rovnice Δx: /we get from the equation Δx:

$$\Delta x = \frac{(log(x)-1)^{2}}{log\,(x)-2} \qquad (7.6)$$

vypočítame limitu /we calculate the limit:

$$\lim_{x\to\infty} \Delta x - \Delta x_{min} = \lim_{x\to\infty} \frac{(log(x)-1)^{2}}{log\,(x)-2} - \left(x - x^{(x-1)/x}\right) = 0 \qquad (7.9)$$

Series expansion at x=∞

$$\frac{1}{log(x)-2} + \frac{log^{2}(x)}{2\,x} + O\left(\left(\frac{1}{x}\right)^{2}\right) \qquad (7.10)$$

Result:

Rovnica (7.9) ukázala, že rovnica $\Delta x \geq \Delta x_{min}$ platí až do nekonečna. / Equation (7.9) showed that equation $\Delta x \geq \Delta x_{min}$ is true to infinity.

Analýza aproximácie / Approximation analysis $\pi(x) = \frac{x}{log\,(x)-\epsilon}$

for

$$\pi(x) = \frac{x}{log\,(x)-\epsilon};\quad \epsilon = 1 + \varepsilon;\quad \varepsilon > 0 \qquad (8.0)$$

Riešením rovnice (8.0) je / The solution of equation (8.0) is:

$$\Delta x = \frac{(x+log(x)+\epsilon)\cdot W_{-1}\left(\frac{\left(e^{\epsilon^{2}/x+\epsilon}\cdot x^{-\epsilon/x}\right)^{x/(x+\epsilon-log\,(x))}\cdot(\epsilon-log(x))}{x-log\,(x)+\varepsilon}\right)}{log\,(x)-\epsilon} + x$$

(8.1)

$W_{-1}(z)\ -\ is\ Lambert\ function,\ reference\ (4)$

V zmysle rovnice (5.5) pre $x > 100$ musí platiť /According to equation (5.5), for $x > 100$, it must hold:

$\Delta x \geq \Delta x_{min}$;

$$\frac{(x+log(x)+\epsilon)\cdot W_{-1}\left(\frac{\left(e^{\epsilon^2/x+\epsilon}\cdot x^{-\epsilon/x}\right)^{x/(x+\epsilon-log\,(x))}\cdot(\epsilon-log(x))}{x-log\,(x)+\epsilon}\right)}{log\,(x)-\epsilon} + x \geq x - x^{(x-1)/x} \qquad (8.2)$$

example(1):

$x = 10^{15}$

$\epsilon = 1.000000000000001$

$\Delta x = \frac{277}{8} = 34.625$

$\Delta x_{min} = 10^{15} - (10^{15})^{(10^{15}-1)/10^{15}} = 34.538...$

$\Delta x > \Delta x_{min}$

example(2):

$x = 10^{16}$

$\epsilon = 1.000000000000001$

$\Delta x = 36$

$\Delta x_{min} = 10^{16} - (10^{16})^{(10^{16}-1)/10^{16}} = 36.841...$

$\Delta x < \Delta x_{min}$

Nie je splnená rovnica 8.2 /Equation 8.2 is not met.

<u>Result analysis:</u>

Rovnica (8.2) pre aproximáciu (8.0) nebude všeobecne splnená. Aproximácia (8.0) je však v zmysle rovnice (3.8) ***väčšia ako*** $\pi(x)$. /Equation (8.2) for the approximation (8.0) will not be generally satisfied. However, the approximation (8.0) ***is greater than*** $\pi(x)$ in the sense of equation (3.8).

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